

ON MULTIPLICATIVELY INDEPENDENT BASES IN CYCLOTOMIC NUMBER FIELDS

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ABSTRACT. Recently the authors [12] showed that the algebraic integers of the form $-m + \zeta_k$ are bases of a canonical number system of $\mathbb{Z}[\zeta_k]$ provided $m \geq \phi(k) + 1$, where ζ_k denotes a k -th primitive root of unity and ϕ is Euler's totient function. In this paper we are interested in the questions whether two bases $-m + \zeta_k$ and $-n + \zeta_k$ are multiplicatively independent. We show the multiplicative independence in case that $0 < |m - n| < 10^6$ and $|m|, |n| > 1$.

1. INTRODUCTION

Let $q \geq 2$ be an integer. Then every positive integer $n \in \mathbb{N}$ has a unique representation of the form

$$n = \sum_{k=0}^{\ell} d_k q^k \quad \text{with} \quad d_k \in \{0, \dots, q-1\}.$$

We call q the base and $\mathcal{D} = \{0, \dots, q-1\}$ the set of digits. This representation can be extended to all integers (positive and negative ones) by taking $q \leq -2$ and $\{0, \dots, |q|-1\}$. In general we call such a pair (q, \mathcal{D}) a numeration system in \mathbb{Z} .

Knuth [8] was one of the first extending the concept of numeration systems to the Gaussian integers by showing that the bases $-1 + i$ and $-1 - i$ together with the set $\{0, 1\}$ form a number system. Later Kátai and Szabó [7] showed that all bases are of the form $-m \pm i$ with set of digits $\mathcal{D} = \{0, 1, \dots, m^2\}$. This was further extended independently to all quadratic number fields by Gilbert [4] and Kátai and Kovács [5, 6].

A different point of view is the following. Let f be a polynomial with coefficients in \mathbb{Z} . Then we call the pair $(X, \{0, \dots, |f(0)|-1\})$ a canonical numeration system in $\mathcal{R} = \mathbb{Z}[X]/(f)$ if every element $g \in \mathbb{Z}[X]/(f)$ has a unique representation of the form

$$g = \sum_{k=0}^{\ell} d_k X^k \quad \text{with} \quad d_k \in \{0, \dots, |f(0)|-1\}.$$

By taking $f = X^2 + 2mX + m^2 + 1$ and $f = X + q$ we get back the numeration systems $(-m \pm i, \{0, 1, \dots, m^2\})$ in $\mathbb{Z}[i]$ and $(-q, \{0, 1, \dots, |q|-1\})$ in \mathbb{Z} respectively.

Let $f = X^d + p_{d-1}X^{d-1} + \dots + p_1X + p_0$ be an irreducible polynomial over \mathbb{Z} with root β , then Kovács [9] could prove that β is a base of a canonical numeration system if $1 \leq p_{d-1} \leq \dots \leq p_1 \leq p_0$ and $p_0 \geq 2$. Pethő [13] replaced the irreducibility condition by supposing that none of the roots of p is a root of unity. Moreover, Kovács [9] showed that for any given order $\mathbb{Z}[\alpha]$ in a number field there exists an element $\beta \in \mathbb{Z}[\alpha]$ such that $(\beta, \{0, 1, \dots, N(\beta)-1\})$ is a numeration system in $\mathbb{Z}[\alpha]$, where $N(\beta)$ denotes the absolute norm of β . Kovács and Pethő [10] gave effectively computable, necessary and sufficient conditions for the element β in order to be the base of a numeration system. The underlying algorithm was improved by Akiyama and Pethő [1] for the special class of polynomials satisfying

$$\sum_{i=1}^d |p_i| < p_0.$$

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2. DEFINITIONS AND STATEMENT OF RESULTS

We want to take a further look at numeration systems in the Gaussian integers. Since $\pm i$ are primitive fourth roots of unity, one may write $-m \pm i$ as $-m + \zeta_4$. Similarly one may see numeration systems in the integers as having bases $-m + \zeta_2 = -(m+1)$. With the above mentioned characterization for quadratic number fields by Gilbert, Kátai and Kovács [4–6] it is easy to show, that algebraic integers of the form $-m + \zeta_3$ and $-m + \zeta_6$ are bases for numeration systems in $\mathbb{Z}[\zeta_3]$ and $\mathbb{Z}[\zeta_6]$, respectively.

In a recent paper [12] the authors extended this to arbitrary cyclotomic number fields.

Theorem 2.1 ([12, Theorem 1.1]). *Let $k > 2$ and m be positive integers and ζ_k be a primitive k -th root of unity. If $m > \varphi(k)$, then $-m + \zeta_k$ is a base of a numeration system in $\mathbb{Z}[\zeta_k]$.*

Their second result in that paper [12] concerns the multiplicative independence of these bases. We call two algebraic integers α and β multiplicatively independent if the equation $\alpha^p = \beta^q$ has only the trivial solution $p = q = 0$ over the integers.

Theorem 2.2 ([12, Theorem 1.2]). *Let $k > 2$ be a positive integer and ζ_k be a primitive k -th root of unity. Then the algebraic integers $-m + \zeta_k$ and $-n + \zeta_k$ are multiplicatively independent provided $m > n > C(k)$, where $C(k)$ is an effectively computable constant depending on k .*

Moreover, if k is a power of 2, 3, 5, 6, 7, 11, 13, 17, 19 or 23, then $-m + \zeta_k$ and $-n + \zeta_k$ are multiplicatively independent as long as $m > n > 1$.

In particular, we conjecture that the second part of the last theorem holds true for all k .

The statement of Theorem 2.2 could be seen as fixing some k and let m and n vary over the positive integers. Therefore the aim of the present paper is to prove the other direction. In particular, we fix the distance of m and n by a and let k vary over the positive integers.

However, before we state our main theorem we need to introduce some further notation. We define the order of a modulo k with $\gcd(a, k) = 1$ as the smallest positive integer n such that $a^n \equiv 1 \pmod k$ and write $n = \text{ord}_k(a)$. Note that in the extremal case $k = 1$ we have $\text{ord}_1(a) = 1$ for all positive integers a . Furthermore, for every prime p we denote by $\nu_p(a)$ the p -adic valuation of an integer a . Then our main result is the following:

Theorem 2.3. *Let $a > 0$ be a given integer and assume that $-m + \zeta_k$ and $-(m + a) + \zeta_k$ with $k > 2$ and m an integer are multiplicatively dependent and none is a root of unity. Furthermore let*

$$S = \{p \in \mathbb{P} : p \mid \Phi_k(m)\Phi_k(m + a)\},$$

where Φ_k is the k -th cyclotomic polynomial. Then, either $m = -1$, $a = 2$ and $k = 4$ or

- $S \subset S_a$, where S_a is the set of prime divisors of a and $S \neq \emptyset$.
- If $k \not\equiv 2 \pmod 4$, then $k \leq \min_{p \in S} \{p^\alpha : p^\alpha \parallel a\}$, $p \nmid k$ and $\text{ord}_k(p) \leq \nu_p(a)$ for all $p \in S$.
- Let $f_p = \text{ord}_k(p)$ for all $p \in S$. Then

$$\Phi_k(x) \prod_{p \in S} p^{\lfloor \nu_p(a)/f_p \rfloor f_p},$$

where either $x = m$ or $x = m + a$. In particular, $f_p \mid \nu_p(\Phi_k(x))$ for all $p \in S$.

Remark 1. We note that without loss of generality we may assume that $k \not\equiv 2 \pmod 4$. Indeed if $k \equiv 2 \pmod 4$ we have $-m + \zeta_k = -m - \zeta_{k/2} = (-1)(-(-m) + \zeta_{k/2})$, where $\zeta_{k/2}$ is a suitable $k/2$ -th primitive root of unity. Therefore $-m + \zeta_k$ and $-(m + a) + \zeta_k$ are multiplicatively independent if and only if $-(-m) + \zeta_{k/2}$ and $-(-m + a) + \zeta_{k/2}$ are.

Remark 2. We want to emphasize that in case that $m \leq 0$ the algebraic integer $-m + \zeta_k$ cannot be a base for $\mathbb{Z}[\zeta_k]$. Therefore in view of canonical number systems only the case $m \geq 1$ is of interest. However for the sake of generality we do not restrict ourselves to the case $m \geq 1$ in Theorem 2.3.

An obvious consequence of Theorem 2.3 is the following Corollary

Corollary 2.4. *For given $a > 0$ there exist at most finitely many pairs $(m, k) \in \mathbb{Z}^+ \times \mathbb{Z}_{\geq 3}$ such that $-m + \zeta_k$ and $-(m + a) + \zeta_k$ are multiplicatively dependent. Moreover the pairs (m, k) are effectively computable.*

Let us remark that applying a result due to Bombieri et. al. [2, Theorem 1] (see also [17, Theorem 3.22]) yields a result similar to Corollary 2.4.

Theorem 2.5 (Bombieri et. al. [2]). *Let \mathcal{C} be a curve defined over $\bar{\mathbb{Q}}$ and let $\phi_1, \dots, \phi_n \in \bar{\mathbb{Q}}(\mathcal{C})$ be multiplicatively independent modulo $\bar{\mathbb{Q}}^*$. Then the set of points $P \in \mathcal{C}(\bar{\mathbb{Q}})$ such that $\phi_1(P), \dots, \phi_n(P)$ are multiplicatively dependent is of bounded height.*

Remark 3. Let $\mathcal{C} \subset \mathbb{A}^2$ be the line $\bar{\mathbb{Q}}[X, Y]/(Y)$ and $\phi_1(X, Y) = X$ and $\phi_2(X, Y) = X + a$, then according to Theorem 2.5 all $x = -m + \zeta_k$ such that $x = -m + \zeta_k$ and $x - a = -(m + a) + \zeta_k$ are multiplicatively dependent are of bounded height, i.e. there exists a constant C_a depending only on a such that $|m| < C_a$. Unfortunately Theorem 2.5 does not provide any information on k . However, let us note that C_a is an effectively computable constant and by dully reproving [17, Theorem 3.22] for this specific instance, but with keeping track of error terms we obtain $C_a = a^{10} + 1$ provided $a > 10$, which yields a much worse bound, than the bound obtained by applying Theorem 2.3.

As mentioned above Theorem 2.3 provides an effective algorithm to determine all pairs (m, k) such that for given a the algebraic integers $-m + \zeta_k$ and $-(m + a) + \zeta_k$ are multiplicatively dependent. Moreover this algorithm is also rather efficient as the next corollary indicates.

Corollary 2.6. *Let $1 \leq a \leq 10^6$ and assume that $m \neq 0, -a$ and $(m, k) \neq (1, 6), (-1, 3), (-a + 1, 6), (-a - 1, 3)$. Then $-m + \zeta_k$ and $-m - a + \zeta_k$ are multiplicatively independent or $m = -1$, $a = 2$ and $k = 4$.*

This result together with the various results in [12] concerning this topic encourages us to state the following conjecture

Conjecture 2.7. *For every $k > 2$ and $m > n > 1$ the algebraic integers $-m + \zeta_k$ and $-n + \zeta_k$ are multiplicatively independent.*

The verification of Corollary 2.6 is based on an algorithm indicated by Theorem 2.3. However, the crucial point is to generate a preferably short list of possible k 's. But the list of possible k 's generated by Theorem 2.3 is rather long for large a . Using such a long list makes it unfeasible to show the multiplicative independence of $-m + \zeta_k$ and $-(m + a) + \zeta_k$ for large a . Therefore instead of using Theorem 2.3 directly we compute for each nonempty $S \subset S_a$ a list of possible k 's, which is in most cases very short or even empty. How to compute such short lists and other computational issues are discussed in Section 4.

3. MULTIPLICATIVE INDEPENDENCE

We may assume that neither $-m + \zeta_k$ nor $-(m + a) + \zeta_k$ are roots of unity and, as indicated by Remark 1, that k is odd or $4|k$.

Let us start with a simple fact about cyclotomic polynomials.

Lemma 3.1. *Assume that $k \not\equiv 2 \pmod{4}$. Then $\Phi_k(x) = \pm 1$ if and only if*

- $x = 0$, or
- $x = 1$ and $k \neq p^\ell$, with a prime p and $\ell > 0$, or
- $x = -1$ and $k \neq 2^\ell$, with $\ell > 1$.

Moreover we have $\Phi_k(1) = p$ if $k = p^\ell$ and $\Phi_{2^\ell}(-1) = 2$, if $\ell > 1$.

Proof. The computation of $\Phi_k(1)$ can be done by Möbius inversion formula applied to

$$k = \lim_{x \rightarrow 1} \frac{x^k - 1}{x - 1} = \prod_{d|k, d \neq 1} \Phi_d(1).$$

See also the proof of the corollary of Theorem 1 in Chapter IV of [11]. In particular this yields $\Phi_k(1) = p$ if $k = p^\ell$ is a prime power and $\Phi_k(1) = 1$ otherwise (see also [3, Proposition 3.5.4]).

Moreover, we know that if k is odd we have $\Phi_k(-x) = \Phi_{2k}(x)$ and $\Phi_{2^\ell}(x) = x^{2^{\ell-1}} + 1$, with $\ell > 0$. Hence all the values of $\Phi_k(\pm 1)$ are as claimed in the lemma provided $k \not\equiv 2 \pmod{4}$.

The computation of $\Phi_k(0) = 1$ is again easy. So we are left to prove that $|\Phi_k(x)| > 1$ for any integer $|x| \geq 2$. But, in this case we have

$$|\Phi_k(x)| = \prod_{\substack{1 \leq a < k \\ \gcd(a, k) = 1}} |x - \zeta_k^a| > \prod_{\substack{1 \leq a < k \\ \gcd(a, k) = 1}} 1.$$

□

In the next step we show that we may exclude the case that both $-m + \zeta_k$ and $-(m + a) + \zeta_k$ are units.

Lemma 3.2. *Assume that $-m + \zeta_k$ and $-(m + a) + \zeta_k$ are both units and multiplicatively dependent. Then one of them is a root of unity. In particular $S \neq \emptyset$ in Theorem 2.3.*

Proof. Since the norm of $-m + \zeta_k$ is $\Phi_k(m)$ we deduce from Lemma 3.1 that $-m + \zeta_k$ is a unit only if $m = 0$ or $m = \pm 1$ and k is not a prime power. Therefore we conclude that $a = 1$ or $a = 2$. However, if $a = 1$ and both $-m + \zeta_k$ and $-m - a + \zeta_k$ are units, then we always obtain that either $m = 0$ or $m + a = 0$, i.e. either $-m + \zeta_k$ or $-m - a + \zeta_k$ is a root of unity.

Therefore we may assume that k is not a prime power and $a = 2$. We have to investigate the Diophantine equation

$$(3.1) \quad (-1 + \zeta_k)^r = (1 + \zeta_k)^s.$$

In particular, we show that if neither $-1 + \zeta_k$ nor $1 + \zeta_k$ is a root of unity, then $r = s = 0$ is the only solution to (3.1). As already mentioned in Remark 1 we may assume that either $4|k$ or k is odd. First, let us assume that k is odd. In this case equation (3.1) can be rewritten as

$$(-1 + \zeta_k)^{r+s} = (-1 + \zeta_k^2)^s.$$

Since we assume that k is odd $-1 + \zeta_k^2$ and $-1 + \zeta_k$ are conjugate and have therefore the same height H . By assumption $-1 + \zeta_k$ is not a root of unity and we have $H > 1$ and $H^{|r+s|} = H^{|s|}$. Therefore either $r = 0$ or $r = -2s$ holds. The first case yields $(1 + \zeta_k)^s = 1$, i.e. $1 + \zeta_k$ is a root of unity or $s = 0$. The second case yields

$$(-1 + \zeta_k)^s (-1 + \zeta_k^2)^s = 1$$

and by taking s -th roots we obtain

$$(3.2) \quad (-1 + \zeta_k)(-1 + \zeta_k^2) = \zeta_k^\ell$$

for some integer ℓ such that ζ_k^ℓ is an s -th root of unity. Let us embed the cyclotomic field $\mathbb{Q}(\zeta_k)$ into \mathbb{C} by $\zeta_k \mapsto \exp(2\pi i/k)$. It is well known that $\exp(ix) = 1 + \theta$ with $|\theta| < |x|$ for real x . Therefore we obtain $|-1 + \exp(4\pi i/k)| < 1/2$ provided $k \geq 26$. On the other hand obviously $|1 + \exp(2\pi i/k)| < 2$, hence the left hand side of equation (3.2) is < 1 in absolute values, i.e. a contradiction. Therefore we may assume that $2 < k \leq 25$ is odd and no prime power, i.e. $k = 15$ or $k = 21$. A direct verification in these two cases shows that equation (3.2) does not hold either.

Let us turn to the case that $4|k$ but k is not a power of 2. Let us write $k = 4n$. In this case we have $-\zeta_k = \zeta_k^{1+2n}$. Since $\gcd(4n, 2n+1) = 1$ we deduce that $(-1 + \zeta_k)$ and $(1 + \zeta_k)$ are conjugate and by the height argument above we obtain that $r = \pm s$. Therefore either $1 - \zeta_k^2$ is a root of unity or $\frac{1+\zeta_k}{1-\zeta_k}$ is a root of unity. Since the equation $1 = x + y$ with $|x| = |y| = 1$ has only sixth roots of unity as solutions we are left to the case that $\frac{1+\zeta_k}{1-\zeta_k}$ is a root of unity. But $|1 + \exp(2\pi i/k)| > 1$ and $|1 - \exp(2\pi i/k)| < 1$ unless $2 < k \leq 6$. But the only integer k in this range divisible by 4 is 4, a prime power of 2. □

According to Theorem 2.3 we suppose that $-m + \zeta_k$ and $-(m + a) + \zeta_k$ are multiplicatively dependent. Then every prime ideal \mathfrak{p} dividing the principal ideal $(-m + \zeta_k)$ also divides the principal ideal $(-(m + a) + \zeta_k)$ and hence the principal ideal (a) . Note that such a prime ideal \mathfrak{p} exists due to Lemma 3.2, hence $p \in S$ where p is the rational prime lying under \mathfrak{p} . Together with Lemma 3.2 this implies the first statement of Theorem 2.3, i.e. $\emptyset \subsetneq S \subset S_a$. Let us fix the prime

ideal \mathfrak{p} and the rational prime p lying under \mathfrak{p} . Computing norms we deduce that some positive power of p divides $\Phi_k(m)$ as well as $\Phi_k(m+a)$.

Next, we aim to prove:

Proposition 3.3. *For an odd prime $p \in S$ we have $p \nmid k$. In case that $p = 2 \in S$ we have either $4 \nmid k$ or $k = 4$, $m = -1$ and $a = 2$.*

Before we state the proof we need the following useful

Lemma 3.4. *Let p be an odd prime with $p|k$ or $p = 2$ and $4|k$, then*

$$\ell = \nu_p(\Phi_k(m)) \leq 1$$

for any integer m .

Proof. The case that $\gcd(m, p) > 1$ is trivial, since in this case $p|m$ and $\Phi_k(m) \equiv 1 \pmod{p}$. Furthermore the case $m = \pm 1$ may be excluded since it is a consequence of Lemma 3.1.

Therefore we may concentrate on the case where $\gcd(m, p) = 1$ and $m \neq \pm 1$. First, let us assume that p is an odd prime. If $q^2|k$ for some prime q , then we know that $\Phi_k(m) = \Phi_{k/q}(m^q)$. Thus we may assume that k is square-free. We write $\Phi_k(m) = p^\ell A$ for some integer A , with $p \nmid A$. Let us note that

$$\Phi_k(x) = \prod_{d|k} (x^d - 1)^{\mu(k/d)}$$

and recall that $p|m^d - 1$ if and only if $\text{ord}_p(m)|d$. If we assume for the moment that $\text{ord}_p(m) \nmid k$, then $m^d - 1$ and p are coprime for all divisors $d|k$ and therefore $\ell = 0$. Now we assume that $\text{ord}_p(m)|k$. Since $\gcd(p, \text{ord}_p(m)) = 1$ there also exists a proper divisor $d|k$ such that $\text{ord}_p(m)|d$. We obtain

$$(3.3) \quad \prod_{\text{ord}_p(m)|d|k} (m^d - 1)^{\mu(k/d)} = p^\ell A \underbrace{\prod_{\text{ord}_p(m) \nmid d|k} (m^d - 1)^{-\mu(k/d)}}_{=: A'}.$$

Note that $\nu_p(A') = 0$, i.e. the p -adic valuation of the right side of (3.3) is ℓ . Let us compute the p -adic valuations on the left side of (3.3). Therefore we use the formula

$$(3.4) \quad \nu_p(m^d - 1) = m_p + \nu_p(d),$$

where for odd primes p we have $m_p = \nu_p(m^{\text{ord}_p(m)} - 1)$. For a proof of formula (3.4) see e.g. [3, Section 2.1.4, Lemma 2.1.22]. Since we assume that k is square-free $\nu_p(m^d - 1) = m_p + 1$ or m_p , depending whether $p|d$ or not. In particular the p -adic valuation on the left side is

$$\underbrace{\sum_{\text{ord}_p(m)|d|k} m_p \mu(k/d)}_{=0} + \sum_{p \nmid \text{ord}_p(m)|d|k} \mu(k/d) = \begin{cases} 1 & \text{if } p \cdot \text{ord}_p(m) = k \\ 0 & \text{else} \end{cases}$$

which proves the lemma for odd primes p .

Now, we turn to the case $p = 2$. Let us assume first that k is not a power of 2. As above we may assume that k is square-free. Moreover, since $\Phi_{2k}(x) = \Phi_k(-x)$ for an odd integer $k > 1$ we may assume that $k > 1$ is odd and square-free (note that we assume that k is not a power of 2). Let us note that $\nu_2(m^d - 1) = \nu_2(m - 1)$, provided d is odd (see e.g. [3, Section 2.1.4, Corollary 2.1.23]). Similarly as in the case $p > 2$ we consider

$$\prod_{d|k} (m^d - 1)^{\mu(k/d)} = 2^\ell A, \quad \text{with } A \text{ odd}$$

and aim to compute ℓ . Since we assume that k is odd also every divisor d of k is odd and we obtain

$$\ell = \nu_2 \left(\prod_{d|k} (m^d - 1)^{\mu(k/d)} \right) = \nu_2(m - 1) \sum_{d|k} \mu(k/d) = 0.$$

We are left with the case that $k = 2^n$ for some $n \geq 2$ and m is odd. By [3, Section 2.1.4, Corollary 2.1.23] we have for even integers $d > 0$ that $\nu_2(m^d - 1) = m_2 + \nu_2(d)$, with

$$m_2 = \begin{cases} \nu_2(m - 1) & \text{if } m \equiv 1 \pmod{4} \\ \nu_2(m^2 - 1) - 1 & \text{else} \end{cases}.$$

Thus we obtain

$$\ell = \nu_2(\Phi_{2^n}(m)) = \nu_2\left(\frac{m^{2^n} - 1}{m^{2^{n-1}} - 1}\right) = m_2 + n - (m_2 + n - 1) = 1.$$

□

Now we return to the

Proof of Proposition 3.3. On the contrary let us assume that p is an odd prime such that $p|k$ or $p = 2$ and $4|k$. Then Lemma 3.4 implies that $\Phi_k(m) = \pm \Phi_k(m + a)$. But this also yields

$$(3.5) \quad -m + \zeta_k = (-(m + a) + \zeta_k)\zeta_k^\ell$$

for some integer ℓ . To show that this is impossible let us note that

$$\left| \frac{\Phi_k(m + a)}{\Phi_k(m)} \right| = \prod_{\substack{1 \leq n \leq k \\ \gcd(n, k) = 1}} \left| \frac{m + a - \zeta_k^n}{m - \zeta_k^n} \right|$$

It is easy to see that $\left| \frac{m + a - \zeta_k^n}{m - \zeta_k^n} \right| > 1$ for all integers $m > \frac{-a+1}{2}$ and $\left| \frac{m + a - \zeta_k^n}{m - \zeta_k^n} \right| < 1$ for all integers $m < \frac{-a-1}{2}$. Therefore we are left with the following two cases:

- a is even and $m = -a/2$, or
- a is odd and $m = \frac{-a+1}{2}$.

First, let us assume that a is even and $p \in S$ is odd. Since $p|a$ and also $p|\frac{a}{2}$ we have

$$\gcd(p, \Phi_k(-a/2)) = 1.$$

The same argument applies if $p = 2$ and $4|a$. Let us assume that $p = 2$ and $a \equiv 2 \pmod{4}$. In this case we have

$$\left| \frac{m + a - \zeta_k}{m - \zeta_k} \right| = \left| \frac{a - 2\zeta_k}{a + 2\zeta_k} \right| \neq 1$$

provided $\operatorname{Re}(\zeta_k) \neq 0$. But $\operatorname{Re}(\zeta_k) = 0$ implies $k = 4$ and by (3.5) we obtain

$$(m + i) = (-m + i)i^\ell,$$

which yields $|m| \leq 1$. And indeed $1 + i$ and $-1 + i$ are multiplicatively dependent, which yields the exception in Proposition 3.3 and Theorem 2.3 respectively.

Let us consider the case that a is odd and $m = \frac{-a+1}{2}$. Note that in this case

$$\left| \frac{m + a - \zeta_k}{m - \zeta_k} \right| = \left| \frac{a + 1 - 2\zeta_k}{-a + 1 - 2\zeta_k} \right| > 1$$

if the real part of ζ_k is larger than $1/2$ and

$$\left| \frac{m + a - \zeta_k}{m - \zeta_k} \right| = \left| \frac{a + 1 - 2\zeta_k}{-a + 1 - 2\zeta_k} \right| < 1$$

if the real part of ζ_k is less than $1/2$. Therefore $\frac{a-1}{2} + \zeta_k$ and $\frac{-a+1}{2} - \zeta_k$ are multiplicatively independent unless $\operatorname{Re}(\zeta_k) = 1/2$. Hence we are left with the case $k = 6$ and due to (3.5) we have to solve the Diophantine equation

$$\frac{a-1}{2} + \zeta_6 = \left(\frac{-a-1}{2} + \zeta_6 \right) \zeta_6^\ell.$$

Solving this equation for $\ell = 0, \dots, 5$ we get either a contradiction or $a = \pm 1, \pm 3$. However in all cases $-m + \zeta_6$ is a root of unity. The case that a is odd and $m = \frac{-a-1}{2}$ runs along the same arguments and therefore we omit it. □

Let us summarize our results so far. Unless $a = 2, m = -1$ and $k = 4$ we have:

- $\emptyset \subsetneq S \subset S_a$
- for all $p \in S$ we have $p \nmid k$.

The following proposition will complete the proof of Theorem 2.3:

Proposition 3.5. *Given the rational prime $p \in S$ and let $n = \nu_p(\gcd(\Phi_k(m), \Phi_k(m+a)))$. Then we have $n \leq \left\lfloor \frac{\nu_p(a)}{f_p} \right\rfloor f_p$ and $f_p | n$, where $f_p = \text{ord}_k(p)$.*

For consistency we continue with the proof of Theorem 2.3 and postpone the proof of Proposition 3.5 to the end of this section.

Note that due to Proposition 3.3 and our assumption that k is odd or $4|k$ we know that $p \nmid k$ and therefore f_p is well-defined.

If $p^{\nu_p(a)} < k$ we obviously have $f_p = \text{ord}_k(p) > \nu_p(a)$. Therefore Proposition 3.5 yields that $0 = n \leq \left\lfloor \frac{\nu_p(a)}{f_p} \right\rfloor f_p$, i.e. $p \notin S$ if we have $p^{\nu_p(a)} < k$. Thus the second statement of Theorem 2.3 is proved, i.e. $k \leq \min_{p \in S} \{p^\alpha : p^\alpha || a\}$. The third statement of Theorem 2.3 is now an immediate consequence of Proposition 3.5. Note that $\Phi_k(m)|\Phi_k(m+a)$ or $\Phi_k(m+a)|\Phi_k(m)$ if $-m + \zeta_k$ and $-(m+a) + \zeta_k$ are multiplicatively dependent. Thus we may assume that $\Phi_k(x) = \gcd(\Phi_k(m), \Phi_k(m+a))$, with $x = m$ or $x = m+a$.

Proof of Proposition 3.5. Let us consider the ideal $I = (-m + \zeta_k, -m - a + \zeta_k)$ in the maximal order $\mathbb{Z}[\zeta_k]$. We note that $N(I) = \gcd(\Phi_k(m), \Phi_k(m+a))$, where $N(I)$ denotes the norm of the ideal I . Moreover we know that $I = (a, -m + \zeta_k)$ and therefore the \mathbb{Z} -module $I' = \langle a, -m + \zeta_k, (-m + \zeta_k)^2, \dots, (-m + \zeta_k)^{\phi(k)-1} \rangle$ is a \mathbb{Z} -submodule of I . Of course I' is also a sublattice in $\mathbb{Z}[\zeta_k]$ and has index

$$[\mathbb{Z}[\zeta_k] : I'] = \det \begin{pmatrix} a & -m & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = a.$$

Since $\mathbb{Z}[\zeta_k] \supset I \supset I'$ we deduce from Lagrange's theorem that $N(I) = [\mathbb{Z}[\zeta_k] : I']a$. Let us fix a rational prime $p \in S$ and let $\mathfrak{p}|(p)$ denote a prime ideal lying above the rational prime p . Then we have

$$\sum_{\mathfrak{p}|(p)} \nu_{\mathfrak{p}}(I) f_{\mathfrak{p}} = \nu_p(N(I)) = n \leq \nu_p(a)$$

where $f_{\mathfrak{p}}$ is the residue class degree of \mathfrak{p} in $\mathbb{Q}(\zeta_k)/\mathbb{Q}$. Let us note that it suffices to show that $f_{\mathfrak{p}} = f_p$ for all prime ideals $\mathfrak{p}|(p)$. Since $\mathbb{Q}(\zeta_k)/\mathbb{Q}$ is a Galois extension all $f_{\mathfrak{p}}$ are equal. Therefore it is enough to show that the residue class degree $f_{\mathfrak{p}}$ of a particular \mathfrak{p} in $\mathbb{Q}(\zeta_k)/\mathbb{Q}$ is exactly $\text{ord}_k(p) = f_p$. However, this is well-known from algebraic number theory and we summarize the crucial facts in the following lemma:

Lemma 3.6. *Let p be a rational prime and let $k > 2$ be an integer such that $p \nmid k$. Then*

- $(p) = \mathfrak{p}^{\phi(p^\ell)}$ is totally ramified in $\mathbb{Z}[\zeta_{p^\ell}]$ and \mathfrak{p} has therefore residue class degree $f_{\mathfrak{p}} = 1$.
- $(p) = \mathfrak{p}_1 \dots \mathfrak{p}_g$ is the prime ideal factorisation in $\mathbb{Z}[\zeta_k]$, where $g = \phi(k)/f_{\mathfrak{p}}$ and $f_{\mathfrak{p}} = \text{ord}_k(p)$ is the residue class degree of any of the prime ideals \mathfrak{p}_i for $i = 1, \dots, g$.

The lemma is proved in most of the algebraic number theory books available, e.g. [14, Chapter 11.3, Lemma N] for the first statement and [16, Theorem 2.13] for the second. \square

4. PROOF OF COROLLARY 2.6

Given an integer a , an application of Theorem 2.3 enables us (at least in principal) to find all pairs (m, k) such that $-m + \zeta_k$ and $-m - a + \zeta_k$ are multiplicatively independent. Finding all pairs (m, k) essentially depends on computing a list of possible k 's. Obviously an integer k satisfying

the conditions of the second point of Theorem 2.3 satisfies

$$k \left| \prod_{\substack{q^\beta < M \\ q \in \mathbb{P} \setminus S}} q^\beta, \right.$$

where $M = \min_{p \in S} \{p^\alpha : p^\alpha \parallel a\}$. But, if a contains only large prime powers the list of possible k 's might get rather long. To overcome this problem we fix a set $\emptyset \subsetneq S \subset S_a$ and compute for this specific subset S a list of possible candidates for k . If a is not too large the number of subsets S usually stays small. In case that $a \leq 10^6$ we know that $|S_a| \leq 7$. Therefore we may assume that S is fixed. In view of Proposition 3.5 we know that for a prime $q < M$ we have $\nu_q(k) \geq \beta$ only if $\text{ord}_{q^\beta}(p) \leq \text{ord}_k(p) \leq \nu_p(a)$ for all $p \in S$. In particular the list of possible k is contained in the list of divisors of

$$K = \prod_{\substack{q < M \\ q \in \mathbb{P} \setminus S}} q^{\beta_q}$$

where β_q is the maximal exponent β such that $\text{ord}_{q^\beta}(p) \leq \nu_p(a)$ for all $p \in S$. The quantity K can be computed rather quickly and we may consider only those k such that $k|K$ and $\text{ord}_k(p) \leq \nu_p(a)$ for all $p \in S$. This approach to compute a list of candidates for k is rather fast and reduces the number of possible k considerably.

In case of a being cube-free there is even a stronger criteria:

Lemma 4.1. *Assume that $\nu_p(a) \leq 2$ for all $p \in S$. Then $k|G$, where G is the gcd of all $p^{\nu_p(a)} - 1$ with $p \in S$.*

Proof. By Proposition 3.5 and the discussion above we have $\text{ord}_{q^\beta}(p) \leq \text{ord}_k(p) \leq 2$ that is $q^\beta|(p-1)$ or $q^\beta|(p+1)$. More precisely for all $p \in S$ with $\nu_p(a) = 1$ we have $q^\beta|(p-1)$ and for all $p \in S$ with $\nu_p(a) = 2$ we have $q^\beta|(p-1)(p+1) = p^2 - 1$. So in any case each $q^\beta|k$ also divides $p^{\nu_p(a)} - 1$ with $p \in S$, i.e. $q^\beta|G$. \square

Let us summarize this by an example. We choose $a = 3^3 * 19 * 127$ and $S = \{3, 19\}$ then we obtain $M = 19$ and $K = 2$, i.e. $k|2$ which we excluded. In case of $S = \{3\}$ we obtain $M = 27$ and $K = 104$ and we have to deal with 5 candidates for $k|K$, namely 4, 8, 13, 52 and 104. Note that $26|104$ but $26 \equiv 2 \pmod{4}$, which we may exclude due to Remark 1. But, only $k = 4, 8, 13$ satisfy $\text{ord}_k(3) \leq 3 = \nu_3(a)$. In case of $S = \{19, 127\}$ we may apply Lemma 4.1 to get $k|\text{gcd}(18, 126) = 18$, hence $k = 3$ or $k = 9$.

Therefore we may assume that both S and k are fixed. By the third statement of Theorem 2.3 we have only finitely many possible values Y for $\Phi_k(x) = Y$ with $x = m$ or $x = m + a$. Solving the polynomial equation $\Phi_k(x) = Y$ for given k and Y we find all possible values for m . However it is far from trivial to find all integral solutions to the equation $\Phi_k(x) = Y$ in case that k is large. Therefore let us describe our approach to this problem.

As already noted at several other places $q^2|k$ implies that $\Phi_k(x) = \Phi_{k/q}(x^q)$ and we also know that $\Phi_{2k}(x) = \Phi_k(x)$ for odd integers k . Therefore we may consider the polynomial equation $\Phi_{k'}(x) = Y$ where k' is the odd-square-free part of k . Since $\Phi_{k'}(0) - Y = 1 - Y$ the possible integral solutions to $\Phi_{k'}(x) = Y$ are $x = \pm d$ with $d|Y - 1$. And it seems easy to check all possible d 's whether they yield a solution to the equation $\Phi_k(x) = Y$ or not. And indeed it is easy as long as the degree of $\Phi_{k'}$ stays considerable small, say $\deg(\Phi_{k'}) = \phi(k') \leq 20000$. In case that $\phi(k') \geq 20000$ computing $\Phi_{k'}(d)$ takes some time and by numerical evidence considering the equation $\Phi_{k'}(x) = Y$ modulo several small primes to exclude a possible solution seems to be more suitable. In particular in case that $\phi(k') \geq 20000$ we check $\Phi_{k'}(d) \equiv Y \pmod{P}$ for all possible solutions d and all primes $P < 10^3$ in order to find a possible solution. In the unlikely case that a possible solution could not be excluded by this approach we verify it directly. Let us note that we chose the threshold 20000 since it seems that for k' with $\phi(k') \geq 20000$ the modulo P approach is faster in our implementation.

For several instances it happened that we found a solution to $\Phi_k(x) = Y$ and therefore found a possible pair (k, m) such that $-m + \zeta_k$ and $-(m \pm a) + \zeta_k$ are multiplicatively dependent. But

if $-m + \zeta_k$ and $-(m \pm a) + \zeta_k$ are multiplicatively dependent, then so are $\Phi_k(m)$ and $\Phi_k(m \pm a)$. Thus there exist integers r, s such that $\Phi_k(m)^r = \Phi_k(m \pm a)^s$. But for a given pair of rational integers it is easy to check whether they are multiplicatively independent or not.

An implementation of all these ideas leads to a rather efficient algorithm (see the Appendix for a concrete implementation). Let us note that the running time varies dramatically with a . For instance the longest runtime for a single instance was 670 seconds obtained by $a = 942479$ while the running time for the instance $a = 950462$ took less than 0.01 seconds. The computation was performed on a standard computer by the computer algebra system SAGE [15] and was split up onto several kernels. The cumulative CPU-computing time was about 36 days and 13 hours.

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